International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1272-1283 www.ijaem.net ISSN: 2395-5252

In the finite element formulation of an Euler-Bernoulli beam element approximating the solution by a polynomial of degree four

Pushpam Kumar Sinha
Department of Mechanical Engineering, NetajiSubhas Institute of Technology, Amhara, Bihta, Patna, Bihar, India

Date of Submission: 01-07-2020 Date of Acceptance: 16-07-2020

ABSTRACT
The governing differential equation for an Euler-Bernoulli beam element is of order four. Hence there are two primary variables, the solution$v(x)$ itself and its derivative$v'(x)$ ($v(x)$ is the transverse deflection of the beam, x is measured along the longitudinal axis of beam with origin at left hand end of beam and positive towards right), appearing in the boundary conditions. This means that for an Euler-Bernoulli beam finite element there are two nodal degrees of freedom. Therefore for a two node finite element there are four unknown quantities to be determined. The best polynomial fit to solution $v(x)$ is cubic polynomial in this case. For this approximation $v''(x)$ is linear. Bending Moment $M(x)$ is proportional to $v''(x)$. As long as $M(x)$ is also linear, the above approximation gives accurate result. However, if in any given region of the beam there is uniformly distributed load (u.d.l.), the $M(x)$ is parabolic which implies that $v''(x)$ must also be a polynomial of degree two. Now if $v''(x)$ is a polynomial of degree two, $v(x)$ must be a polynomial of degree four. Therefore in this paper we present the finite element formulation of Euler-Bernoulli beam element based on polynomial of degree four as the approximation function of the solution $v(x)$ of the governing differential equation. We also demonstrate the accuracy of the method by solving a certain sample problem with u.d.l. over a portion of the beam.

Keywords: Governing differential equation, Shape Function, Polynomial, Weak form Galerkin

I. INTRODUCTION

1.1 A two node Euler-Bernoulli beam element with four degrees of freedom

We begin by formulating weak form Galerkin finite element equation for a 2-node 1-dimensional Euler-Bernoulli beam element. There are several different approaches of finite element method (FEM). And weak form Galerkin is one of them [1,2,3,4,5].

The basic assumption that makes an Euler-Bernoulli beam an Euler-Bernoulli beam is that the plane sections normal to the longitudinal axis of the beam before bending remain plane and normal to the longitudinal axis of the beam after bending. The governing differential equation of Euler-Bernoulli beam is

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) + q(x) = 0 \quad (1)$$

The coordinate system assumed in deriving equation (1) is as shown in Fig.1 for a simply supported beam.

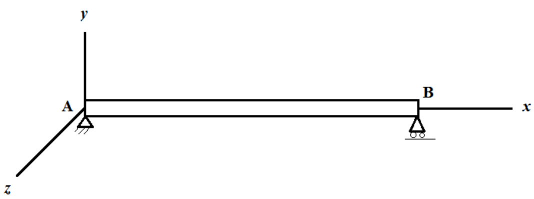

Fig 1: Coordinate system fixed to the beam

In this coordinate system x-axis passes through the centroid of cross-section. In equation (1), E is Young's Modulus of Elasticity of material of the beam at position x, v is transverse deflection of the beam in y direction at position x and $q(x)$ is the intensity of the load at position x. The positive directions of the load intensity $q(x)$, the shear force V and the bending moment M are shown in Fig.2 over an element of the beam [6].

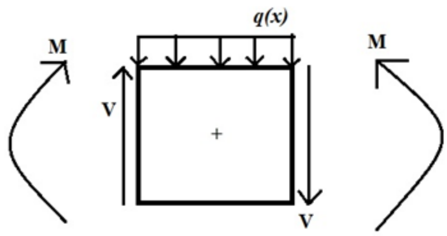

Fig 2: Positive sign convention for*q(x)*, V and M over the beam

Consider that the approximate solution to $v(x)$ in equation (1) is v_a^e for a 2-node 1-dimensional general finite element e. Then the quantity $\frac{d^2}{dx^2}\left(EI\frac{d^2 v_a^e}{dx^2}\right) + q(x)$ is not necessarily zero and this non-zero number is called the residual R of approximation. To find v_a^e we may make this residual R go to zero in a weighted-integral sense, as written below

$$\int_{x_1^e}^{x_2^e} w(x)\left[\frac{d^2}{dx^2}\left(EI\frac{d^2 v_a^e}{dx^2}\right) + q(x)\right] dx = 0 \quad (2)$$

where x_1^e is the position of the left hand end node of the beam element e, x_2^e is the position of the right hand end node of the beam element e, and $w(x)$ is a set of linearly independent functions called the weight functions. In form (2) the approximate solution v_a^e must be differentiable at least four times. To weaken the continuity required of v_a^e, we rewrite equation (2) using integration by parts formula

$$\left[w(x)\frac{d}{dx}\left(EI\frac{d^2 v_a^e}{dx^2}\right)\right]_{x_1^e}^{x_2^e} - \int_{x_1^e}^{x_2^e}\frac{dw(x)}{dx}\frac{d}{dx}\left(EI\frac{d^2 v_a^e}{dx^2}\right) dx + \int_{x_1^e}^{x_2^e} w(x)\, q(x)\, dx = 0$$

Using integration by parts formula for the second term in above equation again, we have

$$\left[w(x)\frac{d}{dx}\left(EI\frac{d^2 v_a^e}{dx^2}\right)\right]_{x_1^e}^{x_2^e} - \left[\frac{dw(x)}{dx}\left(EI\frac{d^2 v_a^e}{dx^2}\right)\right]_{x_1^e}^{x_2^e} + \int_{x_1^e}^{x_2^e}\frac{d^2 w(x)}{dx^2}\left(EI\frac{d^2 v_a^e}{dx^2}\right) dx + \int_{x_1^e}^{x_2^e} w(x)\, q(x)\, dx = 0 \quad (3)$$

There are two boundary terms in equation (3) above. The dependent variable $v(x)$ of differential equation (1) appearing in same form as $w(x)$ in boundary term is primary variable. Therefore for this Euler-Bernoulli beam element there are two primary variables, $v_a^e(x)$ and $\frac{dv_a^e(x)}{dx}$, which means that there are 2 nodal degrees of freedom and therefore for this 2 node Euler-Bernoulli beam element there are four degrees of freedom per element. Therefore the best fit to the approximation function $v_a^e(x)$ over the element is cubic polynomial, as written below

$$v_a^e(x) = a + bx + cx^2 + dx^3 \quad (4)$$

Specification of primary variables at boundary constitutes the Essential Boundary Conditions (EBC). The primary variables at the nodes of a typical Euler-Bernoulli beam element e is shown in Fig. 3.

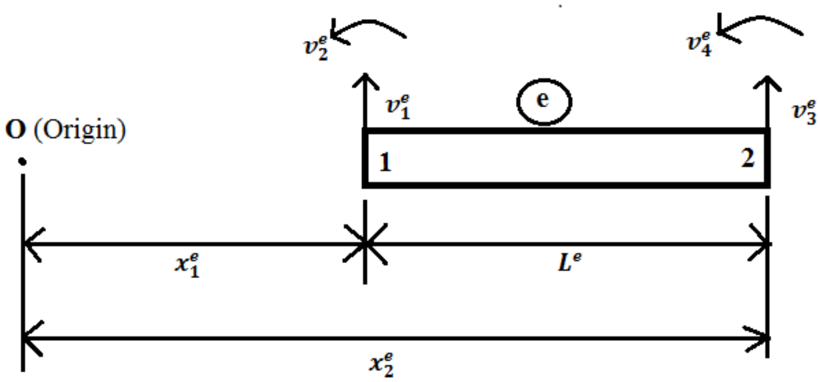

Fig 3: A typical Euler-Bernoulli beam element e with the primary variables shown at nodes

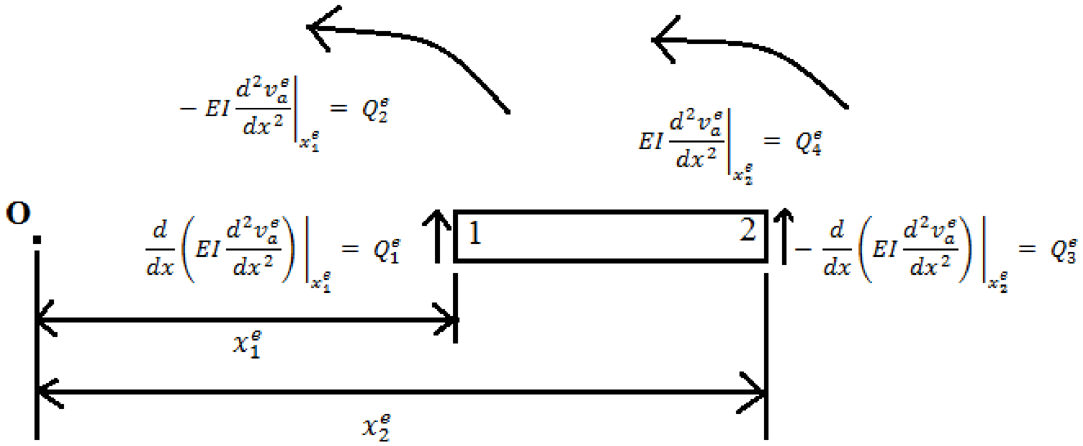

Fig 4: A typical Euler-Bernoulli beam element e with the secondary variables shown at nodes

In Fig. 3, 1 and 2 are the two nodes of element e. And
$v_a^e(x_1^e) = v_1^e, \frac{dv_a^e(x_1^e)}{dx} = v_2^e, v_a^e(x_2^e) = v_3^e, \frac{dv_a^e(x_2^e)}{dx} = v_4^e$(5)

v_1^e, v_2^e, v_3^e and v_4^e are called generalized displacements. Substituting equations (5) in equation (4) and finding constants of equation (4) a, b, c and d in terms of generalized displacements v_1^e, v_2^e, v_3^e and v_4^e we have the following expression for $v_a^e(x)$.

$v_a^e(x) = N_1^e(x)v_1^e + N_2^e(x)v_2^e + N_3^e(x)v_3^e + N_4^e(x)v_4^e$(6)

where
$N_1^e(x) = 1 - \frac{3(x-x_1^e)^2}{{L^e}^2} + \frac{2(x-x_1^e)^3}{{L^e}^3}, N_2^e(x) = (x - x_1^e) - \frac{2(x-x_1^e)^2}{L^e} + \frac{(x-x_1^e)^3}{{L^e}^2}$,(7)
$N_3^e(x) = \frac{3(x-x_1^e)^2}{{L^e}^2} - \frac{2(x-x_1^e)^3}{{L^e}^3}, N_4^e(x) = -\frac{(x-x_1^e)^2}{L^e} + \frac{(x-x_1^e)^3}{{L^e}^2}$
$N_1^e(x), N_2^e(x), N_3^e(x)$ and $N_4^e(x)$ are called shape functions. These are the shape functions in terms of global co-ordinate x. It is clear from equations (6) and (7) that $N_1^e(x_1^e) = 1$ while all the other shape functions at $x = x_1^e$ are zero. Similarly $\frac{d\,N_2^e(x_1^e)}{dx} = 1$ while the first derivative of all the other shape functions at $x = x_1^e$ are zero. Also $N_3^e(x_2^e) = 1$ while all the other shape functions at $x = x_2^e$ are zero, and $\frac{d\,N_4^e(x_2^e)}{dx} = 1$ while the first derivative of all the other shape functions at $x = x_2^e$ are zero .Considering local co-ordinate system in one dimension s, in which origin is fixed at node 1 of the beam element shown in Fig. 3, the relationship between local coordinate s and global coordinate x is

$s = x - x_1^e$(8)

Thus
$s = 0$ at $x = x_1^e$

and
$s = L^e$ at $x = x_2^e$

The shape functions, therefore, in terms of local coordinate s for this beam element are

$N_1^e(s) = 1 - \frac{3\,s^2}{L^{e2}} + \frac{2\,s^3}{L^{e3}}, N_2^e(s) = s - \frac{2\,s^2}{L^e} + \frac{s^3}{L^{e2}}$,(9)
$N_3^e(s) = \frac{3\,s^2}{L^{e2}} - \frac{2\,s^3}{L^{e3}}, N_4^e(s) = -\frac{s^2}{L^e} + \frac{s^3}{L^{e2}}$

CONTINUITY REQUIREMENT ON v_a^e: From equation (3) $v_a^e(x)$ must be at least twice differentiable; therefore the cubic polynomial fit to $v_a^e(x)$ as given by equation (4) or equations (6) and (7) fulfills the continuity requirement on $v_a^e(x)$.

The coefficients of $w(x)$ and $\frac{dw(x)}{dx}$ in boundary terms above in equation (3), i.e. $\frac{d}{dx}\left(EI\frac{d^2v_a^e}{dx^2}\right)$ and $EI\frac{d^2v_a^e}{dx^2}$ are secondary variables. Specification of secondary variables at boundary constitutes the Natural Boundary Conditions (NBC). The secondary variable usually has physical meaning and for the case here $EI\frac{d^2v_a^e}{dx^2}$ is bending moment M and $\frac{d}{dx}\left(EI\frac{d^2v_a^e}{dx^2}\right)$ is shear force V. Therefore, in totality, at nodes 1 and 2 of element e there are 4 NBCs as written below (see Fig. 4)

$$\frac{d}{dx}\left(EI\frac{d^2v_a^e}{dx^2}\right)\Big|_{x_1^e} = Q_1^e \quad (10a)$$

$$-EI\frac{d^2v_a^e}{dx^2}\Big|_{x_1^e} = Q_2^e \quad (10b)$$

$$-\frac{d}{dx}\left(EI\frac{d^2v_a^e}{dx^2}\right)\Big|_{x_2^e} = Q_3^e \quad (10c)$$

$$EI\frac{d^2v_a^e}{dx^2}\Big|_{x_2^e} = Q_4^e \quad (10d)$$

Q_1^e, Q_2^e, Q_3^e and Q_4^e are called generalized forces. The positive value for term $\frac{d}{dx}\left(EI\frac{d^2v_a^e}{dx^2}\right)$ on right hand end of beam element stands for downward direction for shear force V (see Fig. 2). However, for beam element for the purposes of making finite element calculations we take upward direction of shear force at both ends positive (see Fig. 4). Hence negative sign is placed in equation (10c) on left hand side. Similarly the positive value for term $EI\frac{d^2v_a^e}{dx^2}$ on left hand end of beam element stands for clockwise sense for bending moment M (see Fig. 2). However, for beam element for the purposes of making finite element calculations we take anticlockwise sense of bending moment at both ends positive (see Fig. 4). Hence negative sign is placed in equation (10b) on left hand side.

With equations (10a) to (10d) for generalized forces, we have from equation (3)

$$\int_{x_1^e}^{x_2^e} EI\,\frac{d^2w(x)}{dx^2}\frac{d^2v_a^e}{dx^2}dx - w(x_1^e)Q_1^e - \frac{dw}{dx}(x_1^e)Q_2^e - w(x_2^e)Q_3^e - \frac{dw}{dx}(x_2^e)Q_4^e + \int_{x_1^e}^{x_2^e} w(x)\,q(x)\,dx = 0 \quad (11)$$

or $I_1 - B_1 + I_2 = 0$ (12)

where

$$I_1 = \int_{x_1^e}^{x_2^e} EI\,\frac{d^2w(x)}{dx^2}\frac{d^2v_a^e}{dx^2}dx \quad (13a)$$

$$B_1 = w(x_1^e)Q_1^e + \frac{dw}{dx}(x_1^e)Q_2^e + w(x_2^e)Q_3^e + \frac{dw}{dx}(x_2^e)Q_4^e \quad (13b)$$

$$I_2 = \int_{x_1^e}^{x_2^e} w(x)\,q(x)\,dx \quad (13c)$$

The Galerkin's approach for this finite element formulation states that

$$w(x) = N_j^e(x), j = 1,2,3,4 \quad (14)$$

From equation (3) or equation (11) we see that $w(x)$ must be at least twice differentiable. With the choice of $w(x)$ as given in equation (14), the continuity requirement on $w(x)$ is fulfilled. We begin calculations for I_1 by taking $w(x) = N_1^e(x)$ first.

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1272-1283 www.ijaem.net ISSN: 2395-5252

CALCULATING I_1:
CASE 1: $w(x) = N_1^e(x)$

$I_1 = \int_{x_1^e}^{x_2^e} EI \frac{d^2}{dx^2} N_1^e(x) \frac{d^2}{dx^2}[N_1^e(x)v_1^e + N_2^e(x)v_2^e +$ N3exv3e+ N4exv4e dx(15)

But
$N_1^e(x) = N_1^e(x - x_1^e), N_2^e(x) = N_2^e(x - x_1^e),$
$N_3^e(x) = N_3^e(x - x_1^e), N_4^e(x) = N_4^e(x - x_1^e)$

Also $x - x_1^e = s$, the local coordinate (see equation (8)). Therefore the entire equation (15) can be recast in terms of local coordinate s as

$I_1 = \int_0^{L^e} EI \frac{d^2}{ds^2} N_1^e(s) \frac{d^2}{ds^2}[N_1^e(s)v_1^e + N_2^e(s)v_2^e +$ N3esv3e+ N4esv4e ds(16)

Assume that $EI$is constant throughout the length L^e of finite element e. Then

$$I_1 = EI \int_0^{L^e} \left(\frac{12\,s}{L^{e3}} - \frac{6}{L^{e2}}\right)\left[\left(\frac{12\,s}{L^{e3}} - \frac{6}{L^{e2}}\right)v_1^e + \left(\frac{6\,s}{L^{e2}} - \frac{4}{L^e}\right)v_2^e + \left(\frac{6}{L^{e2}} - \frac{12s}{L^{e3}}\right)v_3^e + \left(\frac{6\,s}{L^{e2}} - \frac{2}{L^e}\right)v_4^e\right] ds$$

or$I_1 = EI\left(\frac{12}{L^{e3}}v_1^e + \frac{6}{L^{e2}}v_2^e - \frac{12}{L^{e3}}v_3^e + \frac{6}{L^{e2}}v_4^e\right)$

$$\text{or} I_1 = EI\begin{bmatrix}\frac{12}{L^{e3}} & \frac{6}{L^{e2}} & -\frac{12}{L^{e3}} & \frac{6}{L^{e2}}\end{bmatrix}\begin{bmatrix}v_1^e\\v_2^e\\v_3^e\\v_4^e\end{bmatrix} \text{(17a)}$$

CASE 2: $w(x) = N_2^e(x)$

$$I_1 = EI\begin{bmatrix}\frac{6}{L^{e2}} & \frac{4}{L^e} & -\frac{6}{L^{e2}} & \frac{2}{L^e}\end{bmatrix}\begin{bmatrix}v_1^e\\v_2^e\\v_3^e\\v_4^e\end{bmatrix} \text{(17b)}$$

CASE 3: $w(x) = N_3^e(x)$

$$I_1 = EI\begin{bmatrix}-\frac{12}{L^{e3}} & -\frac{6}{L^{e2}} & \frac{12}{L^{e3}} & -\frac{6}{L^{e2}}\end{bmatrix}\begin{bmatrix}v_1^e\\v_2^e\\v_3^e\\v_4^e\end{bmatrix} \text{(17c)}$$

CASE 4: $w(x) = N_4^e(x)$

$$I_1 = EI\begin{bmatrix}\frac{6}{L^{e2}} & \frac{2}{L^e} & -\frac{6}{L^{e2}} & \frac{4}{L^e}\end{bmatrix}\begin{bmatrix}v_1^e\\v_2^e\\v_3^e\\v_4^e\end{bmatrix} \text{(17d)}$$

CALCULATING B_1:
When$w(x) = N_1^e(x), B_1 = Q_1^e$(18a)
When$w(x) = N_2^e(x), B_1 = Q_2^e$(18b)
When$w(x) = N_3^e(x), B_1 = Q_3^e$(18c)
When$w(x) = N_4^e(x), B_1 = Q_4^e$(18d)

CALCULATING I_2:
Assume that $q(x)$ is constant over a certain finite element e and given by
$q(x) = q_e$ (say) (19)
CASE 1: $w(x) = N_1^e(x)$

$$I_2 = \int_0^{L^e} N_1^e(s)q_e\, ds$$

or$I_2 = \frac{q_e L^e}{2}$(20a)
CASE 2: $w(x) = N_2^e(x)$

$$I_2 = \int_0^{L^e} N_2^e(s)q_e\, ds$$

or$I_2 = \frac{q_e L^{e2}}{12}$(20b)
CASE 3: $w(x) = N_3^e(x)$

$$I_2 = \int_0^{L^e} N_3^e(s)q_e\, ds$$

or$I_2 = \frac{q_e L^e}{2}$(20c)
CASE 4: $w(x) = N_4^e(x)$

$$I_2 = \int_0^{L^e} N_4^e(s)q_e\, ds$$

or$I_2 = -\frac{q_e L^{e2}}{12}$(20d)

Using results of equations (17a) to (17d), (18a) to (18d), and (20a) to (20d); we write equation (12) for all the four different choices of $w(x)$ for the said beam element into one single matrix equation as

$$EI\begin{bmatrix}\frac{12}{L^{e3}} & \frac{6}{L^{e2}} & -\frac{12}{L^{e3}} & \frac{6}{L^{e2}}\\ \frac{6}{L^{e2}} & \frac{4}{L^e} & -\frac{6}{L^{e2}} & \frac{2}{L^e}\\ -\frac{12}{L^{e3}} & -\frac{6}{L^{e2}} & \frac{12}{L^{e3}} & -\frac{6}{L^{e2}}\\ \frac{6}{L^{e2}} & \frac{2}{L^e} & -\frac{6}{L^{e2}} & \frac{4}{L^e}\end{bmatrix}\begin{bmatrix}v_1^e\\v_2^e\\v_3^e\\v_4^e\end{bmatrix} = \begin{bmatrix}Q_1^e\\Q_2^e\\Q_3^e\\Q_4^e\end{bmatrix} - \frac{q_e L^e}{2}\begin{bmatrix}1\\L^e/6\\1\\-L^e/6\end{bmatrix} \text{(21)}$$

This is the weak form Galerkin finite element equation for the Euler-Bernoulli beam element e.
1.2 A sample problem and its solution by the method formulated in section 1.1

PROBLEM: A cantilever beam of uniform I cross-section of length 1m is loaded as shown in Fig.5. The cross-sectional dimensions of the beam are $b = 60\,mm, t = 8\,mm, h = 120\,mm,$ and $h_1 = 100\,mm$ (see Fig. 6). Find the deflection, slope and stresses in the beam using FEM.. Assume the Young's modulus of elasticity of the beam as $E = 200\ GPa$.

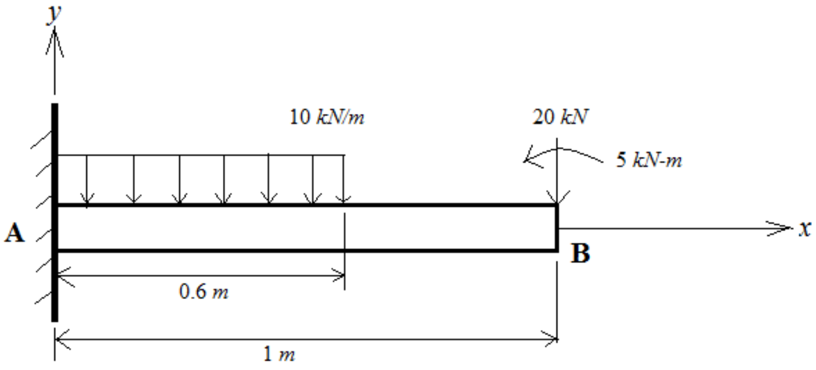

Fig 5: A cantilever beam

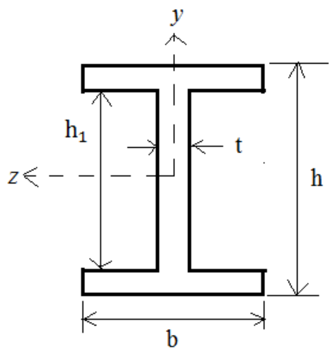

Fig 6: The cross-section of the beam shown in Fig. 5

SOLUTION: Moment of inertia of beam about z axis (see Fig. 6)

$$I = I_{zz} = \frac{1}{12}[b(h^3 - h_1^3) + th_1^3]$$

or $I = 4.306666667\text{x}\ 10^{-6} m^4$

$EI = 8.6133333\text{x}10^5\ Nm^2$

Discretize the domain $(0 < x < 1\ m)$ into a minimum of two finite elements: $(0 < x < 0.6\ m)$ and $(o.6\ m < x < 1\ m)$. The global degrees of freedom of the beam are shown in Fig. 7. And the element nodal degrees of freedom are shown in Fig. 8.

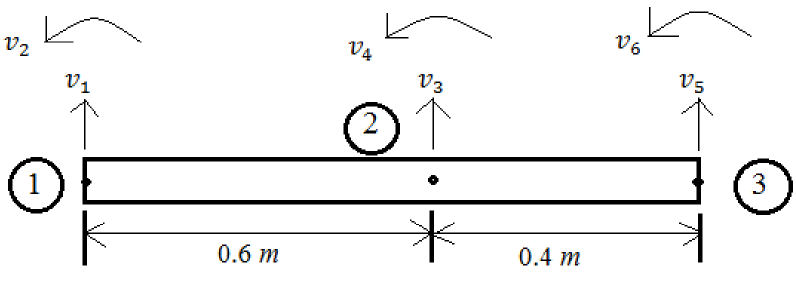

Fig 7: Global degrees of freedom of the beam

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1272-1283 www.ijaem.net ISSN: 2395-5252

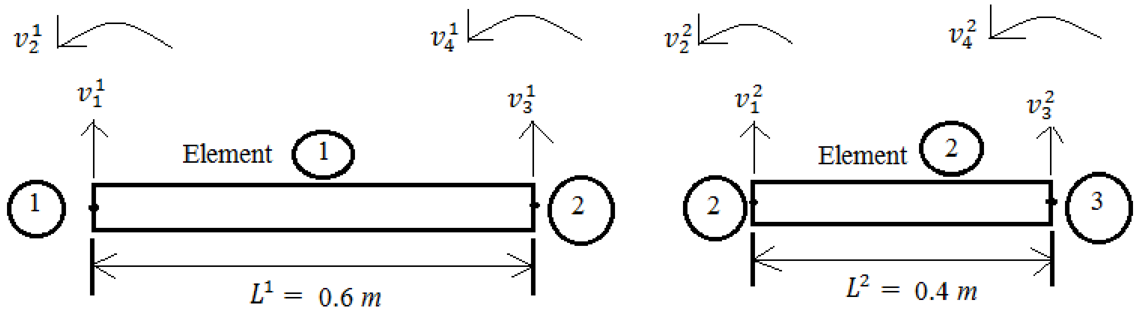

Fig 8: Element nodal degrees of freedom of the two finite elements 1 and 2

The beam has three nodes 1, 2 and 3 as shown in Fig.7 and Fig.8. Also from figures 7 and 8 we relate the element nodal degrees of freedom and global degrees of freedom as

$v_1^1 = v_1, v_2^1 = v_2, v_3^1 = v_1^2 = v_3, v_4^1 = v_2^2 = v_4,$
$v_3^2 = v_5, v_4^2 = v_6$

Using equation (21) to write the finite element equations of finite elements 1 and 2 respectively, assembling the two equations together as one single matrix equation for the whole domain, and imposing the boundary conditions, and thereafter solving the four linear simultaneous equations in as many unknowns we have the following solution to this problem (the process of assembly and the process of deciding boundary conditions is carried out as explained in reference [1])

$v_3 = -24.88276\text{x } 10^{-4}\ m, v_4 = -66.914\text{x } 10^{-4}\ rad,$
$v_5 = -51.962\text{x } 10^{-4}\ m, v_6 = -62.289957\text{x } 10^{-4}\ rad$

The approximate solution to the deflection v, v_a^1 for element 1 of the beam is

$v_A^1(s) = N_3^1(s)v_3 + N_4^1(s)v_4$
or $v_a^1 = 44.5237037\text{x } 10^{-4}s^3 - 95.833\text{x } 10^{-4}s^2$ (26)

The approximate solution to the deflection v, v_a^2 for element 2 of the beam is

$$v_A^2(s) = N_1^2(s)v_3 + N_2^2(s)v_4 + N_3^2(s)v_5 + N_4^2(s)v_6$$

or $v_a^2 = 38.7\text{x } 10^{-4}s^3 - 17.44\text{x } 10^{-4}s^2 - 66.914\text{x}10^{-4}\ s - 24.88276\text{x } 10^{-4}$ (27)

The stress σ^1 in element 1 of the beam is

$$\sigma^1 = -yE\frac{d^2v_a^1}{ds^2}$$
$$= -20\ y(267.14222\ s - 191.666)\ MPa$$

$(\sigma^1)_{Top\ fibre} = -320.57066\ s + 229.9992\ MPa$ (28)

$(\sigma^1)_{Top\ fibre}\ (s = 0) \cong 230\ MPa, (\sigma^1)_{Top\ fibre}\ (s = 0.6) = 37.6568\ MPa$ (29)

The stress σ^2 in element 2 of the beam is

$$\sigma^2 = -yE\frac{d^2v_a^2}{ds^2}$$
$$= -20\ y(232.2\ s - 34.88)\ MPa$$

$(\sigma^2)_{Top\ fibre} = -278.64\ s + 41.856\ MPa$ (30)

$(\sigma^2)_{Top\ fibre}\ (s = 0) = 41.856\ MPa, (\sigma^2)_{Top\ fibre}\ (s = 0.4) = -69.6\ MPa$ (31)

Note that

$(\sigma^1)_{Top\ fibre}\ (s = 0.6) \neq (\sigma^2)_{Top\ fibre}\ (s = 0)$

This is so because we had not ensured continuity of $\frac{d^2v}{ds^2}$ at inter-element node. From free body diagram of the beam or a portion of the beam and corresponding equations of equilibrium we know that the moment at node 1, M_1 is -16.8 kN-m, and the moment at node 2, M_2 is -3 kN-m. We already know from given loading conditions of the beam that the moment at node 3, M_3 is 5 kN-m. Therefore the stresses σ_1, σ_2 and σ_3 in the top fibre of the beam at nodes 1,2 and 3 respectively are

$$\sigma_1 = \frac{-0.06\ M_1}{I} = 234\ MPa, \sigma_2 = \frac{-0.06\ M_2}{I} = 41.786\ MPa, \sigma_3 = \frac{-0.06\ M_3}{I} = -69.6\ MPa \quad (32)$$

We note that

$(\sigma^2)_{Top\ fibre}\ (s = 0) \cong \sigma_2, (\sigma^2)_{Top\ fibre}\ (s = 0.4 = \sigma 3$

But

$(\sigma^1)_{Top\ fibre}\ (s = 0.6) \neq \sigma_2$ and $(\sigma^1)_{Top\ fibre}\ (s = 0) \neq \sigma_1$

This discrepancy is due to two facts

(a) We had not ensured continuity of $\frac{d^2v}{ds^2}$ at inter-element node.

(b) The variation of bending moment in the region of the beam where uniformly distributed load acts is parabolic where as in this case $\frac{d^2v_a^1(s)}{ds^2}$ is linear. This means that by taking a polynomial approximation to deflection v of the beam of higher degree (greater than three here) , we can achieve accuracy.

In this paper we approximate v by a polynomial of degree four in the region of the beam where uniformly distributed load acts.

II. A THREE NODE EULER-BERNOULLI BEAM ELEMENT WITH FIVE DEGREES OF FREEDOM

Consider the three node Euler-Bernoulli beam element e with five degrees of freedom as shown in Fig.9. Node 3 is internal to the element and mid-way between nodes 1 and 2.

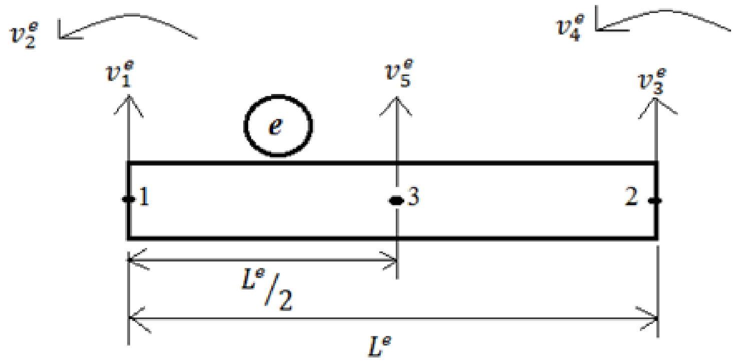

Fig 9: The 3 node Euler-Bernoulli beam element e with 5 degrees of freedom

As seen earlier in section 1.1 there are two primary variables for the Euler-Bernoulli beam element which means that there are 2 nodal degrees of freedom, and therefore for the 3 node Euler-Bernoulli beam element there must be six degrees of freedom per element. However, in the case as shown in Fig.9 we consider the deflection v_5^e at node 3 as the only degree of freedom at that node. This consideration is with an important assumption that no concentrated load and no concentrated moment acts at the internal node 3.

The best fit to approximation function $v_a^e(s)$ over such an element is a polynomial of degree four because only such a polynomial has five unknown constants to be determined in terms of five generalized displacements $v_1^e, v_2^e, v_3^e, v_4^e$ and v_5^e. Note that s is the local coordinate as discussed earlier in section 1.1. Let us derive $v_a^e(s)$ now.

$v_a^e(s) = a_1 + a_2 s + a_3 s^2 + a_4 s^3 + a_5 s^4$ (33a)

a_1, a_2, a_3, a_4, a_5 are constants

$\frac{d\, v_a^e(s)}{ds} = a_2 + 2a_3 s + 3a_4 s^2 + 4a_5 s^3$ (33b)

We know that (from Fig.9)

at $s = 0, v_a^e = v_1^e, \frac{dv_a^e}{ds} = v_2^e,$

at $s = L^e/2, v_a^e = v_5^e$

and at $s = L^e, v_a^e = v_3^e, \frac{dv_a^e}{ds} = v_4^e$

Putting these in equations (33a), (33b), and solving for a_1, a_2, a_3, a_4, a_5 we have for $v_a^e(s)$

$v_a^e(s) = N_1^e(s)v_1^e + N_2^e(s)v_2^e + N_5^e(s)v_5^e + N_3^e(s)v_3^e + N_4^e(s)v_4^e$ (34)

where $N_1^e(s), N_2^e(s), N_5^e(s), N_3^e(s), N_4^e(s)$ are shape functions given by

$N_1^e(s) = 1 - \frac{11\,s^2}{{L^e}^2} + \frac{18\,s^3}{{L^e}^3} - \frac{8\,s^4}{{L^e}^4}, N_2^e(s) = s - \frac{4\,s^2}{L^e} + \frac{5\,s^3}{{L^e}^2} - \frac{2\,s^4}{{L^e}^3},$

$N_5^e(s) = \frac{16\,s^2}{{L^e}^2} - \frac{32\,s^3}{{L^e}^3} + \frac{16\,s^4}{{L^e}^4}, N_3^e(s) = -\frac{5\,s^2}{{L^e}^2} + \frac{14\,s^3}{{L^e}^3} - \frac{8\,s^4}{{L^e}^4},$ (35)

$$N_4^e(s) = \frac{s^2}{L^e} - \frac{3\,s^3}{{L^e}^2} + \frac{2\,s^4}{{L^e}^3}$$

III. WEAK FORM GALERKIN FINITE ELEMENT FORMULATION OF THE ELEMENT DISCUSSED IN SECTION 2

The weak form of the governing differential equation (1) for the beam element discussed in section 3 in terms of local coordinate s is

$$\int_0^{L^e} EI \frac{d^2 w}{ds^2}\frac{d^2 v_a^e}{ds^2} ds - w(0)Q_1^e - \frac{dw}{ds}(0)Q_2^e$$
$$-w(L^e/2)Q_5^e - \frac{dw}{ds}(L^e/2)Q_6^e$$
$$- w(L^e)Q_3^e - \frac{dw}{ds}(L^e)Q_4^e$$
$$+ \int_0^{L^e} w(s)\, q(s)\, ds = 0 \quad (36)$$

where

$$Q_1^e = \frac{d}{ds}\left(EI\frac{d^2 v_a^e}{ds^2}\right)\Big|_0, Q_2^e = -EI\frac{d^2 v_a^e}{ds^2}\Big|_0, Q_3^e = -\frac{d}{ds}\left(EI\frac{d^2 v_a^e}{ds^2}\right)\Big|_{L^e}, Q_4^e = EI\frac{d^2 v_a^e}{ds^2}\Big|_{L^e},$$

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1272-1283 www.ijaem.net ISSN: 2395-5252

$$Q_5^e = \left[-\frac{d}{ds}\left(EI\frac{d^2 v_a^e}{ds^2}\right)\Big|_{\frac{L^e}{2}^-} + \frac{d}{ds}\left(EI\frac{d^2 v_a^e}{ds^2}\right)\Big|_{\frac{L^e}{2}^+} \right], (37)$$

$$Q_6^e = \left[\left(EI\frac{d^2 v_a^e}{ds^2}\Big|_{\frac{L^e}{2}^-} \right) + \left(-EI\frac{d^2 v_a^e}{ds^2}\Big|_{\frac{L^e}{2}^+} \right) \right]$$

Also Q_5^e =Externally applied concentrated load at node 3 = 0 (for this case)
and Q_6^e =Externally applied concentrated moment at node 3 = 0 (for this case)
Equation (36) can be re-written as

$$I_1 - B_1 + I_2 = 0 (38)$$

where

$$I_1 = \int_0^{L^e} EI \frac{d^2 w}{ds^2}\frac{d^2 v_a^e}{ds^2} ds (39a)$$

$$B_1 = w(0)Q_1^e + \frac{dw}{ds}(0)Q_2^e + w(L^e)Q_3^e + \frac{dw}{ds}(L^e)Q_4^e (39b)$$

$$I_2 = \int_0^{L^e} w(s)\, q(s)\, ds (39c)$$

To apply the Galerkin's approach to the weak form (38) we take

$$w(s) = N_j^e(s), j = 1,2,3,4,5$$
(40)

Calculating I_1, B_1, I_2 for the different forms of $w(s)$ separately one by one, as stated in equation (40), we get the following weak form Galerkin finite element equation for the beam element discussed in section 2

$$EI\begin{bmatrix} \frac{63.2}{L^{e^3}} & \frac{33.8}{L^{e^2}} & \frac{-102.4}{L^{e^3}} & \frac{39.2}{L^{e^3}} & \frac{-6.8}{L^{e^2}} \\ \frac{33.8}{L^{e^2}} & \frac{7.2}{L^e} & \frac{-25.6}{L^{e^2}} & \frac{6.8}{L^{e^2}} & \frac{-1.2}{L^e} \\ \frac{-102.4}{L^{e^3}} & \frac{-25.6}{L^{e^2}} & \frac{204.8}{L^{e^3}} & \frac{-102.4}{L^{e^3}} & \frac{25.6}{L^{e^2}} \\ \frac{39.2}{L^{e^3}} & \frac{6.8}{L^{e^2}} & \frac{-102.4}{L^{e^3}} & \frac{63.2}{L^{e^3}} & \frac{-18.8}{L^{e^2}} \\ \frac{-6.8}{L^{e^2}} & \frac{-1.2}{L^e} & \frac{25.6}{L^{e^2}} & \frac{-18.8}{L^{e^2}} & \frac{7.2}{L^e} \end{bmatrix} \begin{bmatrix} v_1^e \\ v_2^e \\ v_5^e \\ v_3^e \\ v_4^e \end{bmatrix} = \begin{bmatrix} Q_1^e \\ Q_2^e \\ 0 \\ Q_3^e \\ Q_4^e \end{bmatrix} + q_e L^e \begin{bmatrix} -7/30 \\ -L^e/60 \\ -8/15 \\ -7/30 \\ L^e/60 \end{bmatrix} (41)$$

In deriving equation (41) it is assumed that EI is constant over the length L^e of the beam element and $q(s)$ also is constant equal to q_e over the length L^e of the beam element e.

IV. SAMPLE PROBLEM OF SECTION 1.2 BUT WITH SOLUTION BASED ON FORMULATION IN SECTION 3

Discretize the domain $(0 < x < 1\,m)$ into a minimum of two finite elements: $(0 < x < 0.6\,m)$ and $(o.6\,m < x < 1\,m)$. The global degrees of freedom of the beam are shown in Fig. 10. And the element nodal degrees of freedom are shown in Fig. 11. It is obvious from Fig.11 that the finite element 1 is a 3 node Euler-Bernoulli beam element, where as the finite element 2 is a 2 node Euler-Bernoulli beam element. Hence, the formulation derived in section 3 applies to the finite element 1, where as that derived in section 1.1 applies to the finite element 2. From figures 10 and 11 we relate element nodal degrees of freedom and global degrees of freedom as

$$v_1^1 = v_1, v_2^1 = v_2, v_3^1 = v_1^2 = v_3, v_4^1 = v_2^2 = v_4, v_5^1 = v_5, v_3^2 = v_6, v_4^2 = v_7$$

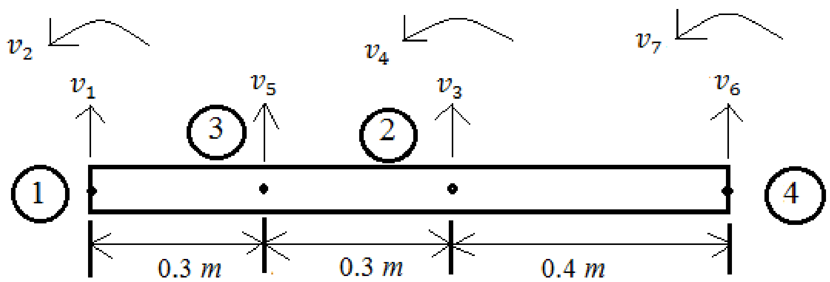

Fig 10: Global degrees of freedom of the beam

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 1, pp: 1272-1283 www.ijaem.net ISSN: 2395-5252

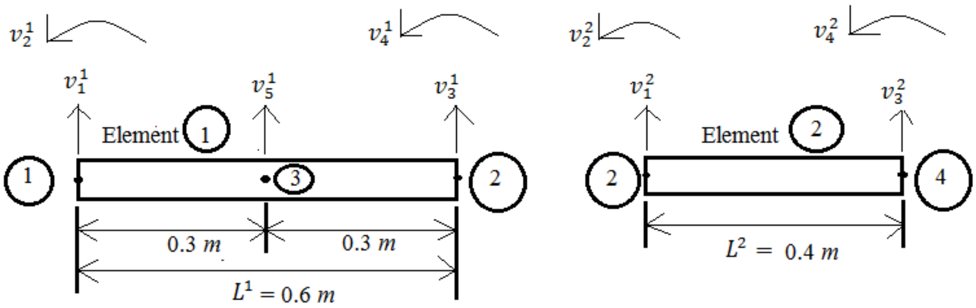

Fig 11: Element nodal degrees of freedom of the two finite elements 1 and 2

The finite element equation of element 1 is (putting $L^1 = 0.6\ m$ and $q_1 = 10000\ N/m$ in equation (41))

$$EI\begin{bmatrix} 292.59 & 93.89 & -474.07 & 181.48 & -18.89 \\ 93.89 & 12 & -71.11 & 18.89 & -2 \\ -474.07 & -71.11 & 948.15 & -474.07 & 71.11 \\ 181.48 & 18.89 & -474.07 & 292.59 & -52.22 \\ -18.89 & -2 & 71.11 & -52.22 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_5 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} Q_1^1 - 1400 \\ Q_2^1 - 60 \\ -3200 \\ Q_3^1 - 1400 \\ Q_4^1 + 60 \end{bmatrix} (42)$$

The finite element equation of element 2 is (putting $L^2 = 0.4\ m$ and $q_2 = 0$ in equation (21))

$$EI\begin{bmatrix} 187.5 & 37.5 & -187.5 & 37.5 \\ 37.5 & 10 & -37.5 & 5 \\ -187.5 & -37.5 & 187.5 & -37.5 \\ 37.5 & 5 & -37.5 & 10 \end{bmatrix}\begin{bmatrix} v_3 \\ v_4 \\ v_6 \\ v_7 \end{bmatrix} = \begin{bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{bmatrix} (43)$$

The boundary conditions are

$v_1 = 0, v_2 = 0$(44a)

$Q_3^1 + Q_1^2$ =Externally applied concentrated load at node 2 $= 0$(44b)

$Q_4^1 + Q_2^2$ =Externally applied concentrated moment at node 2$= 0$(44c)

$Q_3^2 = -20000, Q_4^2 = 5000$(44d)

Assembling element equations (42) and (43) together into one single matrix equation and using equations (44a) to (44d), we have

$$EI\begin{bmatrix} 292.59 & 93.89 & -474.07 & 181.48 & -18.89 & 0 & 0 \\ 93.89 & 12 & -71.11 & 18.89 & -2 & 0 & 0 \\ -474.07 & -71.11 & 948.15 & -474.07 & 71.11 & 0 & 0 \\ 181.48 & 18.89 & -474.07 & 480.09 & -14.72 & -187.5 & 37.5 \\ -18.89 & -2 & 71.11 & -14.72 & 22 & -37.5 & 5 \\ 0 & 0 & 0 & -187.5 & -37.5 & 187.5 & -37.5 \\ 0 & 0 & 0 & 37.5 & 5 & -37.5 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_5 \\ v_3 \\ v_4 \\ v_6 \\ v_7 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1^1 - 1400 \\ Q_2^1 - 60 \\ -3200 \\ -1400 \\ 60 \\ -20000 \\ 5000 \end{bmatrix} (45)$$

From rows 3,4,5,6 and 7 respectively of the equation (45) we have five linear simultaneous equations in as many unknowns; solving them (by, say, Gauss elimination method) we have

$v_5 = -7.45\text{x } 10^{-4}\ m,\ v_3 = -24.85\text{x } 10^{-4}\ m,\ v_4 = -66.82\text{x } 10^{-4}\ rad,$

$v_6 = -51.89\text{x } 10^{-4}\ m,\ v_7 = -62.18\text{x } 10^{-4}\ rad$

The approximate solution to the deflection v, $v_a^1$for element 1 of the beam is

$v_A^1(s) = N_5^1(s)v_5 + N_3^1(s)v_3 + N_4^1(s)v_4$

or$v_a^1 = -4.50617\text{x } 10^{-4}s^4 + 49.88888\text{x } 10^{-4}s^3 - 97.33889\text{x } 10^{-4}s^2$(46a)

The stress σ^1in element 1 of the beam is

$$\sigma^1 = -yE\,\frac{d^2v_a^1}{ds^2}$$

$= -20\, y(-54.07404\, s^2 + 299.33328\, s - 194.67778)\ MPa$

$(\sigma^1)_{Top\ fibre} = 64.88885\, s^2 - 359.19994\, s + 233.61334\ MPa$(47)

$(\sigma^1)_{Top\ fibre}\ (s = 0) \cong 234\ MPa, (\sigma^1)_{Top\ fibre}\ (s = 0.6) \cong 41.8\ MPa$ (48)

We note that $v_3, v_4, v_6$and $v_7$calculated in this section is almost same as $v_3, v_4, v_5$and $v_6$respectively calculated in the section 1.2. Therefore, the stress σ^2in element 2 of the beam in top fibre is

$(\sigma^2)_{Top\ fibre} = -278.64\, s + 41.856\ MPa$(49)

$(\sigma^2)_{Top\ fibre}\ (s = 0) \cong 41.8\ MPa, (\sigma^2)_{Top\ fibre}\ (s = 0.4) = -69.6\ MPa$ (50)

We note that in this formulation we get nearly correct values for $(\sigma^1)_{Top\ fibre}\ (s = 0)$, and $(\sigma^1)_{Top\ fibre}\ (s = 0.6$, as given by equations (32).

V. CONCLUSIONS AND DISCUSSION

Approximating the solution of governing differential equation (1) by an interpolating polynomial of degree four in the region of the beam where uniformly distributed load (u.d.l.) acts gives accurate result for the sample problem treated by FEM in this paper. Notable in my work is that I have considered the partial degrees of freedom (five instead of six for a 3 node Euler-Bernoulli beam element), where as, the weak form of equation (1) advocated for the presence of two nodal degrees of freedom. Though I neglected one nodal degree of freedom at the internal node, the formulation gave correct result for the sample problem discussed in this paper. Additionally, to keep things simple, I had assumed that the external concentrated load and the external concentrated moment at the internal node of 3 node Euler-Bernoulli beam element are zero. To get an idea of the complication involved otherwise, define $\boldsymbol{B_1}$from equation (36) as

$$B_1 = w(0)Q_1^e + \frac{dw}{ds}(0)Q_2^e + w(L^e/2)Q_5^e + \frac{dw}{ds}(L^e/2)Q_6^e + w(L^e)Q_3^e + \frac{dw}{ds}(L^e)Q_4^e$$

When$\boldsymbol{w(s) = N_1^e(s)}$,

$$B_1 = Q_1^e - \frac{3\, Q_6^e}{2\, L^e}$$

We get corresponding results for $w(s) = N_j^e(s), j = 2,5,3,4$

We had, in this paper, solved the sample problem by weak form Galerkin Finite Element Formulation. However, this is not the only approach of FEM by which we can treat the Euler-Bernoulli beam. For local elasticity, which is the case here for the sample problem discussed, the principle of minimum total potential energy also can be used to derive the finite element formulation [7]. For nonlocal elasticity theory, discussed in paper [7], the principle of minimum total potential energy cannot be used; instead the weak from Galerkin approach is the better alternative. In local elasticity, the stress at a point can be uniquely written in terms of the strain at that point, but in non-local elasticity it cannot be done so.

In [8], Sanjay Kumar has compared analytical solution and FEM solution of cantilever beam subjected to u.d.l. and v.d.l. (varying distributed load), and he confesses that the FEM solution is slightly different from the analytical because the interpolating polynomial, to solution, assumed by the FEM was a cubic polynomial. In this paper I had proposed a remedy to this very problem.

In both my work and work [8], the beam considered was cantilever beam. However, in [9], Gunakala S.R. et al proposed the Finite Element solution for simply supported and clamped beam under uniform load. But they did not point out the same inconsistency in result as pointed by me here and by Sanjay Kumar in [8].

REFERENCES

[1]. Bathe K-J. (2014). Finite Element Procedures. Prentice Hall, Pearson Education, Inc.. Watertown, MA, USA.

[2]. Desai YM, Eldho TI, and Shah AH. (2011). Finite Element Method with Applications in Engineering. Pearson, Noida.

[3]. Prudhomme S, and Cottereau R. (Fall 2016). An Introduction to the Finite Element Method with ComsolMultiphysics. Lecture Notes MTH8207. EcolePolytechnique De Montreal.

[4]. Rao SS. (2011). The Finite Element Method in Engineering. Butterworth-Heinemann. MA01803, USA.

[5]. Reddy JN. (2005). An Introduction to the Finite Element Method. Tata McGraw-Hill Publishing Company Limited. New Delhi.

[6]. Gere JM, and Goodno BJ. (2009). Mechanics of Materials. Cengage Learning India Private Limited. Delhi.

[7]. Phadikar JK, and Pradhan SC. (2010). Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates. Computational Materials Science; 49:492-499.

[8]. Kumar S. (2016). Comparison of deflection and slope of cantilever beam with Analytical

and Finite Element Method for different loading conditions. International Journal of Engineering Science and Innovative Technology; Vol. 5, Issue 6:45-51.

[9]. Gunakala SR, et al. (2012). A Finite Element Solution of the beam equation via MATLAB. International Journal of Applied Science and Technology; Vol. 2, No. 8:80-88.

International Journal of Advances in Engineering and Management

ISSN: 2395-5252

IJAEM

Volume: 02 Issue: 01 DOI: 10.35629/5252

www.ijaem.net

Email id: ijaem.paper@gmail.com